

A Lewy-Stampacchia Estimate for quasilinear variational inequalities in the Heisenberg group

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Abstract

We consider an obstacle problem in the Heisenberg group framework, and we prove that the operator on the obstacle bounds pointwise the operator on the solution. More explicitly, if $\varepsilon \geq 0$ and \bar{u}_ε minimizes the functional

$$\int_{\Omega} (\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{p/2}$$

among the functions with prescribed Dirichlet boundary condition that stay below a smooth obstacle ψ , then

$$\begin{aligned} 0 &\leq \operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \\ &\leq \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+. \end{aligned}$$

Introduction

In this paper, we extend the so called dual estimate of [10] to the obstacle problem for quasilinear elliptic equations in the Heisenberg group.

The notation we use is the standard one: for $n \geq 1$, we consider \mathbb{R}^{2n+1} endowed with the group law

$$\begin{aligned} &(x^{(1)}, y^{(1)}, t^{(1)}) \circ (x^{(2)}, y^{(2)}, t^{(2)}) \\ &:= \left(x^{(1)} + x^{(2)}, y^{(1)} + y^{(2)}, t^{(1)} + t^{(2)} + 2(x^{(2)} \cdot y^{(1)} - x^{(1)} \cdot y^{(2)}) \right), \end{aligned}$$

for any $(x^{(1)}, y^{(1)}, t^{(1)}), (x^{(2)}, y^{(2)}, t^{(2)}) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, where the “ \cdot ” is the standard Euclidean scalar product.

Then, we denote by \mathbb{H}^n the n -dimensional Heisenberg group, i.e., \mathbb{R}^{2n+1} endowed with this group law.

The coordinates are usually written as $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, and, as customary, we introduce the left invariant vector fields (X, Y) induced by the group law

$$X_j := \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t} \text{ and } Y_j := \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t},$$

for $j = 1, \dots, n$, and the horizontal gradient $\nabla_{\mathbb{H}^n} := (X, Y)$. The main issue of the Heisenberg group is that X and Y do not commute, that is

$$[X, Y] = 4 \frac{\partial}{\partial t} \neq 0.$$

We are interested in studying the obstacle problem in this framework. For this, we consider a smooth function $\psi : \mathbb{H}^n \rightarrow \mathbb{R}$, which will be our obstacle (more precisely, ψ is supposed to have continuous derivatives of second order in X and Y).

Fixed a bounded open set Ω with smooth boundary, and $p \in (1, +\infty)$, we consider the space $W_{\mathbb{H}^n}^{1,p}(\Omega)$ to be the set of all functions u in $L^p(\Omega)$ whose distributional horizontal derivatives $X_j u$ and $Y_j u$ belong to $L^p(\Omega)$, for $j = 1, \dots, n$.

Such space is naturally endowed with the norm

$$\|u\|_{W_{\mathbb{H}^n}^{1,p}(\Omega)} := \|u\|_{L^p(\Omega)} + \sum_{j=1}^n \left(\|X_j u\|_{L^p(\Omega)} + \|Y_j u\|_{L^p(\Omega)} \right).$$

We call $W_{\mathbb{H}^n,0}^{1,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ with respect to this norm.

We fix a smooth domain $\Omega_\star \ni \Omega$, $u_\star \in W_{\mathbb{H}^n}^{1,p}(\Omega_\star) \cap L^\infty(\Omega_\star)$ and we introduce the space

$$\mathcal{K} := \{u \in W_{\mathbb{H}^n}^{1,p}(\Omega) \text{ s.t. } u \leq \psi, \text{ and } u - u_\star \in W_{\mathbb{H}^n,0}^{1,p}(\Omega)\}.$$

Loosely speaking, \mathcal{K} is the space of all the functions having prescribed Dirichlet boundary datum equal to u_\star along $\partial\Omega$ and that stay below the obstacle ψ .

Now we consider a parameter $\varepsilon \geq 0$ and we deal with the variational problem

$$\inf_{u \in \mathcal{K}} \mathcal{F}_\varepsilon(u; \Omega), \text{ where } \mathcal{F}_\varepsilon(u; \Omega) := \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{p/2}. \quad (1)$$

By direct methods, it is seen that such infimum is attained (see, e.g., the compactness result in [18, 5] or references therein) and so we consider a minimizer \bar{u}_ε .

Then, \bar{u}_ε is a solution of the variational inequality¹

$$\int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n} (v - \bar{u}_\varepsilon) \geq 0, \quad (2)$$

¹Formula (2) may be easily obtained this way. Fixed $v \in W_{\mathbb{H}^n}^{1,p}(\Omega)$ with $v \leq \psi$, and $v - \bar{u}_\varepsilon \in W_{\mathbb{H}^n,0}^{1,p}(\Omega)$, for any $t \geq 0$, let $u^{(t)} := \bar{u}_\varepsilon + t(v - \bar{u}_\varepsilon)$. Notice that

$$u^{(t)} := (1-t)\bar{u}_\varepsilon + tv \leq (1-t)\psi + t\psi \leq \psi,$$

hence $u^{(t)} \in \mathcal{K}$. So, by the minimality of \bar{u}_ε , we have $\mathcal{F}_\varepsilon(u^{(0)}; \Omega) = \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; \Omega) \leq \mathcal{F}_\varepsilon(u^{(t)}; \Omega)$ for any $t \geq 0$. Consequently,

$$\begin{aligned} 0 &\leq \lim_{t \searrow 0} \frac{\mathcal{F}_\varepsilon(u^{(t)}; \Omega) - \mathcal{F}_\varepsilon(u^{(0)}; \Omega)}{t} \\ &= \int_\Omega (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot \nabla_{\mathbb{H}^n} (v - \bar{u}_\varepsilon), \end{aligned}$$

that is (2).

for any $v \in W_{\mathbb{H}^n}^{1,p}(\Omega)$ with $v \leq \psi$, and $v - \bar{u}_\varepsilon \in W_{\mathbb{H}^n,0}^{1,p}(\Omega)$.

These kind of variational inequalities has now receiving a considerable attention (see, e.g., [6] and references therein), even when $p = 2$ (notice that in such a case ε does not play any role). We observe that, when $p \neq 2$, the operator driving the variational inequality in (2) is not linear anymore (in fact, it may be seen as the Heisenberg group version of the p -Laplace operator): for these kind of operators even the regularity theory is more problematic than expected at a first glance, and many basic fundamental questions are still open (see, e.g., [7], [11], [12] and [19]): this is a crucial difference with respect to the Euclidean case, so we think it is worth dealing with the problem in such a generality.

Now, we introduce the set of p 's for which our main result holds. The definition we give is slightly technical, but, roughly speaking, consists in taking the set of all the p 's for which a pointwise bound for the operator of a sequence of minimal solutions is stable under uniform limit. The further difficulty of taking this assumption is due to the lack of a thoroughgoing regularity theory for the quasilinear Heisenberg equation (as remarked in Lemma 12 at the end of this paper, this technicality may be skipped when $p = 2$).

Definition 1. Let $p \in (1, +\infty)$. We say that $p \in \mathcal{P}(\psi, \Omega)$ if the following property holds true:

For any $\varepsilon > 0$, any $v \in W_{\mathbb{H}^n}^{1,p}(\Omega)$, any $M > 0$, any sequence $F_k = F_k(r, \xi) \in C([-M, M] \times \Omega)$, with $F_k(\cdot, \xi) \in C^1([-M, M])$ and

$$0 \leq \partial_r F_k \leq \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+, \quad (3)$$

if $u_k : \Omega \rightarrow [-M, M]$ is a sequence of minimizers of the functional

$$\int_{\Omega} \frac{1}{p} (\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_k(u(\xi), \xi) d\xi \quad (4)$$

over the functions $u \in W_{\mathbb{H}^n}^{1,p}(\Omega)$, $u - v \in W_{\mathbb{H}^n,0}^{1,p}(\Omega)$, with the property that u_k converges to some u_∞ uniformly in Ω , we have that

$$\begin{aligned} 0 &\leq \operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} u_\infty|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\infty \right) \\ &\leq \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+. \end{aligned} \quad (5)$$

As remarked² in Lemma 12 at the end of this paper, we always have that

$$2 \in \mathcal{P}(\psi, \Omega).$$

In particular, the main result of this paper (i.e., the forthcoming Theorem 2) always holds for $p = 2$ without any further restriction. We think that it is an interesting open problem to decide whether or not other values of p belong

²As usual, the superscript “+” denotes the positive part of a function, i.e. $f^+(x) := \max\{f(x), 0\}$.

to $\mathcal{P}(\psi, \Omega)$, in general, or at least when the right hand side of (5) is close to zero (e.g., when the obstacle is almost flat). For instance, the property in Definition 1 would be satisfied in presence of a Hölder regularity theory for the horizontal gradient for solutions of quasilinear equations in the Heisenberg group – namely, if one knew that bounded solutions of $\operatorname{div}_{\mathbb{H}^n} ((\varepsilon + |\nabla_{\mathbb{H}^n} u|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u) = f$, with f bounded, have Hölder continuous horizontal gradient, with interior estimates (this would be the Heisenberg counterpart of classical regularity results for the Euclidean case, see, e.g., Theorem 1 in [17]; notice also that it would be a regularity theory for the equation, not for the obstacle problem). As far as we know, such a theory has not been developed yet, not even for minimal solutions (see, however, [3, 12, 19] for the case of homogeneous equations).

The result we prove here is:

Theorem 2. *If $p \in \mathcal{P}(\psi, \Omega)$ then*

$$\begin{aligned} 0 &\leq \operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \\ &\leq \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+. \end{aligned} \quad (6)$$

The result in Theorem 2 is quite intuitive: when \bar{u}_ε does not touch the obstacle, it is free to make the operator vanish. When it touches and sticks to it, the operator is driven by the one of the obstacle – and on these touching points the obstacle has to bend in a somewhat convex fashion, which justifies the first inequality in (6) and superscript “+” in the right hand side of (6).

Figure 1, in which the thick curve represents the touching between \bar{u}_ε and the obstacle, tries to describe this phenomena. On the other hand, the set in which \bar{u}_ε touches the obstacle may be very wild, so the actual proof of Theorem 2 needs to be more technical than this.

In fact, the first inequality of (6) is quite obvious since it follows, for instance, by taking $v := \bar{u}_\varepsilon - \varphi$ in (2), with an arbitrary $\varphi \in C_0^\infty(\Omega, [0, +\infty))$, so the core of (6) lies on the second inequality: nevertheless, we think it is useful to write (6) in this way to emphasize a control from both the sides of the operator applied to the solution.

We remark that the right hand side of (6) is always finite when $\varepsilon > 0$, and when $\varepsilon = 0$ and $p \geq 2$. In this case, (6) is an L^∞ -bound and may be seen as a regularity result for the solution of the obstacle problem. It is worth noticing that such regularity result holds for $\varepsilon = 0$ as well, only assuming that $p \in \mathcal{P}(\psi, \Omega)$, which is a requirement on the problem when $\varepsilon > 0$.

On the other hand, if $\varepsilon = 0$ and $p \in (1, 2)$, the right hand side of (6) may become infinite (in this case (6) is true, but meaningless, stating that something is less than or equal to an infinite quantity).

In the Euclidean setting, the analogue of (6) was first obtained in [10] for the Laplacian case, and it is therefore often referred to with the name of Lewy-Stampacchia Estimate. It is also called Dual Estimate, for it is, in a sense, obtained by the duality expressed by the variational inequality (2). Other Authors refer to it with the name of Penalization Method, for the role played by ε .

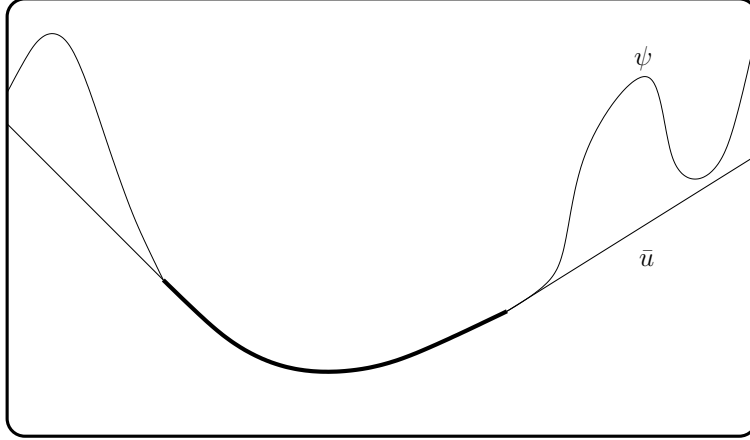


Figure 1: *Touching the obstacle*

After [10], estimates of these type became very popular and underwent many important extensions and strengthenings: see, among the others, [15, 14, 8, 1, 13]. As far as we know, the estimate we prove is new in the Heisenberg group setting, even for $p = 2$.

Hereafter, we deal with the proof of Theorem 2. First, in § 1, we prove Theorem 2 when $\varepsilon > 0$.

The proof when $\varepsilon = 0$ is contained in § 3 and it is based on a limit argument, i.e., we consider the problem with $\varepsilon > 0$, we use Theorem 2 in such a case, and then we pass $\varepsilon \searrow 0$. This procedure is quite delicate though, because, as far as we know, it is not clear whether or not the Heisenberg group setting allows a complete Hölder regularity theory for first derivatives (see [7]). To get around this point, in § 2, we study the L^p -convergence of the solution \bar{u}_ε of the ε -problem to the solution \bar{u}_0 of the problem with $\varepsilon = 0$, which, we believe, is of independent interest (see, in particular Propositions 3 and 4).

The paper ends with an Appendix that collects some ancillary results needed in § 2.

1 Proof of Theorem 2 when $\varepsilon > 0$

We prove (6) in the simpler case $\varepsilon > 0$ (the case $\varepsilon = 0$ will be dealt with in § 3). The technique used in this proof is a variation of a classical penalized test function method (see, e.g., [15, 14, 8, 1, 13] and references therein), and several steps of this proof are inspired by some estimates obtained by [4] in the Euclidean case.

First of all, we set

$$\mu := -1 + \min \left\{ \inf_{\bar{\Omega}} \psi, \inf_{\bar{\Omega}} u_* \right\} \in \mathbb{R}$$

and we observe that

$$\bar{u}_\varepsilon \geq \mu \quad (7)$$

a.e. in Ω . Indeed, let $w := \max\{\bar{u}_\varepsilon, \mu\}$. Since ψ and u_\star are below μ in Ω , we have that $w \in \mathcal{K}$, thus

$$0 \leq \mathcal{F}_\varepsilon(w; \Omega) - \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; \Omega) = - \int_{\Omega \cap \{\bar{u}_\varepsilon < \mu\}} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \leq 0,$$

and, from this, (7) plainly follows.

Now, let $\eta \in (0, 1)$, to be taken arbitrarily small in the sequel. Let also

$$h := \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+. \quad (8)$$

Notice that

$$\|h\|_{L^\infty(\Omega)} < +\infty, \quad (9)$$

because $\varepsilon > 0$. For any $t \in \mathbb{R}$, we consider the truncation function

$$H_\eta(t) := \begin{cases} 0 & \text{if } t \leq 0, \\ t/\eta & \text{if } 0 < t < \eta, \\ 1 & \text{if } t \geq \eta. \end{cases}$$

Now, we take u_η to be a weak solution of

$$\begin{cases} \operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \right) = h \cdot (1 - H_\eta(\psi - u_\eta)) & \text{in } \Omega, \\ u_\eta = \bar{u}_\varepsilon & \text{on } \partial\Omega. \end{cases} \quad (10)$$

where, as usual, the boundary datum is attained in the trace sense: such a function u_η may be obtained by the direct method in the calculus of variations, by minimizing the functional

$$\int_\Omega \frac{1}{p} (\varepsilon + |\nabla_{\mathbb{H}^n} u(\xi)|^2)^{p/2} + F_\eta(u(\xi), \xi) d\xi$$

over $u \in W_{\mathbb{H}^n}^{1,p}(\Omega)$, $u - \bar{u}_\varepsilon \in W_{\mathbb{H}^n,0}^{1,p}(\Omega)$, where

$$F_\eta(r, \xi) := \int_0^r h(\xi) \cdot (1 - H_\eta(\psi(\xi) - \sigma)) d\sigma.$$

Now, we claim that

$$u_\eta \leq \psi \text{ a.e. in } \Omega. \quad (11)$$

To establish this, we use the test function $(u_\eta - \psi)^+$ in (10). Since, on $\partial\Omega$, we have $(u_\eta - \psi)^+ = (\bar{u}_\varepsilon - \psi)^+ = 0$, we obtain that

$$\begin{aligned} & - \int_\Omega \left((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \right) \cdot \nabla_{\mathbb{H}^n} (u_\eta - \psi)^+ \\ &= \int_\Omega h \cdot (1 - H_\eta(\psi - u_\eta)) (u_\eta - \psi)^+ = \int_\Omega h \cdot (u_\eta - \psi)^+. \end{aligned}$$

Consequently, by (8),

$$\begin{aligned}
& \int_{\Omega} \left[\left((\varepsilon + |\nabla_{\mathbb{H}^n} u_{\eta}|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_{\eta} \right) \right. \\
& \quad \left. - \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right] \cdot \nabla_{\mathbb{H}^n} (u_{\eta} - \psi)^+ \\
&= \int_{\Omega} \left[\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) - h \right] \cdot (u_{\eta} - \psi)^+ \\
&\leq 0.
\end{aligned}$$

By the strict monotonicity of the operator (i.e., by the strict convexity of the function $\mathbb{R}^{2n} \ni \zeta \mapsto (\varepsilon + |\zeta|^2)^{p/2}$), it follows that $(u_{\eta} - \psi)^+$ vanishes almost everywhere in Ω , proving (11).

Now, we claim that

$$\bar{u}_{\varepsilon} \geq u_{\eta} \text{ a.e. in } \Omega. \quad (12)$$

To verify this, we consider the test function

$$\tau := \bar{u}_{\varepsilon} + (u_{\eta} - \bar{u}_{\varepsilon})^+.$$

We notice that

$$\tau = \begin{cases} \bar{u}_{\varepsilon} & \text{in } \{u_{\eta} \leq \bar{u}_{\varepsilon}\}, \\ u_{\eta} & \text{in } \{u_{\eta} > \bar{u}_{\varepsilon}\}, \end{cases}$$

hence $\tau \leq \psi$, due to (11). Furthermore, on $\partial\Omega$, we have that $\tau = \bar{u}_{\varepsilon}$, due to the boundary datum in (10). Therefore the obstacle problem variational inequality (2) gives that

$$\begin{aligned}
0 &\leq \int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_{\varepsilon}|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_{\varepsilon} \right) \cdot \nabla_{\mathbb{H}^n} (\tau - \bar{u}_{\varepsilon}) \\
&= \int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_{\varepsilon}|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_{\varepsilon} \right) \cdot \nabla_{\mathbb{H}^n} (u_{\eta} - \bar{u}_{\varepsilon})^+.
\end{aligned} \quad (13)$$

On the other hand, from (10),

$$\begin{aligned}
& \int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} u_{\eta}|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_{\eta} \right) \cdot \nabla_{\mathbb{H}^n} (u_{\eta} - \bar{u}_{\varepsilon})^+ \\
&= - \int_{\Omega} h \cdot (1 - H_{\eta}(\psi - u_{\eta})) \cdot (u_{\eta} - \bar{u}_{\varepsilon})^+ \leq 0.
\end{aligned} \quad (14)$$

By (13) and (14), we obtain that

$$\begin{aligned}
& \int_{\Omega} \left[\left((\varepsilon + |\nabla_{\mathbb{H}^n} u_{\eta}|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_{\eta} \right) \right. \\
& \quad \left. - \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_{\varepsilon}|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_{\varepsilon} \right) \right] \cdot \nabla_{\mathbb{H}^n} (u_{\eta} - \bar{u}_{\varepsilon})^+ \leq 0.
\end{aligned}$$

This and the strict monotonicity of the operator implies that $(u_{\eta} - \bar{u}_{\varepsilon})^+$ vanishes almost everywhere in Ω , hence proving (12).

Now, we claim that

$$\bar{u}_\varepsilon \leq u_\eta + \eta \text{ in } \Omega. \quad (15)$$

To do this, we set

$$\theta := \bar{u}_\varepsilon - (\bar{u}_\varepsilon - u_\eta - \eta)^+.$$

Notice that $\theta \leq \bar{u}_\varepsilon \leq \psi$, and also that, on $\partial\Omega$, $\theta = \bar{u}_\varepsilon$. As a consequence, (2) gives that

$$\begin{aligned} 0 &\leq \int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \cdot \nabla_{\mathbb{H}^n} (\theta - \bar{u}_\varepsilon) \\ &= - \int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \cdot \nabla_{\mathbb{H}^n} (\bar{u}_\varepsilon - u_\eta - \eta)^+. \end{aligned} \quad (16)$$

On the other hand, $(\bar{u}_\varepsilon - u_\eta - \eta)^+ = 0$ on $\partial\Omega$, and

$$\begin{aligned} \{\bar{u}_\varepsilon - u_\eta - \eta > 0\} &\subseteq \{\psi - u_\eta > \eta\} \\ &\subseteq \{1 - H_\eta(\psi - u_\eta) = 0\}, \end{aligned}$$

and therefore, by (10),

$$\begin{aligned} &\int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} (u_\eta + \eta)|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} (u_\eta + \eta) \right) \cdot \nabla_{\mathbb{H}^n} (\bar{u}_\varepsilon - u_\eta - \eta)^+ \\ &= \int_{\Omega} \left((\varepsilon + |\nabla_{\mathbb{H}^n} u_\eta|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} u_\eta \right) \cdot \nabla_{\mathbb{H}^n} (\bar{u}_\varepsilon - u_\eta - \eta)^+ \\ &= - \int_{\Omega} h \cdot (1 - H_\eta(\psi - u_\eta)) \cdot (\bar{u}_\varepsilon - u_\eta - \eta)^+ = 0. \end{aligned} \quad (17)$$

Then, (16) and (17) yield that

$$\begin{aligned} &\int_{\Omega} \left[\left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \right. \\ &\quad \left. - \left((\varepsilon + |\nabla_{\mathbb{H}^n} (u_\eta + \eta)|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} (u_\eta + \eta) \right) \right] \cdot \nabla_{\mathbb{H}^n} (\bar{u}_\varepsilon - u_\eta - \eta)^+ \\ &\leq 0. \end{aligned}$$

Thus, in this case, the strict monotonicity of the operator implies that $(\bar{u}_\varepsilon - u_\eta - \eta)^+$ vanishes almost everywhere in Ω , and so (15) is established.

In particular, by (11), (15) and (7),

$$\|u_\eta\|_{L^\infty(\Omega)} \leq 2 + \|\psi\|_{L^\infty(\Omega)} + \|u_\star\|_{L^\infty(\Omega)}. \quad (18)$$

Moreover, by (12) and (15), we have that

$$u_\eta \text{ converges uniformly in } \Omega \text{ to } \bar{u}_\varepsilon \quad (19)$$

as $\eta \searrow 0$.

Furthermore

$$0 \leq \partial_r F_\eta(r, \xi) \leq h(\xi) = \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+$$

hence (6) follows³ from (19) and the fact that $p \in \mathcal{P}(\psi, \Omega)$ (recall (5) in Definition 1). \square

2 Estimating the L^p -distance between $\nabla_{\mathbb{H}^n} \bar{u}_0$ and $\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$

The purpose of this section is to consider the solution \bar{u}_ε of the ε -problem and the solution \bar{u}_0 of the problem with $\varepsilon = 0$, and to bound the L^p -norm of $|\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|$. Such estimate is quite technical and it is different according to the cases $p \in (1, 2]$ and $p \in [2, +\infty)$: see the forthcoming Propositions 3 and 4.

As a matter of fact, we think that the estimates proved in Propositions 3 and 4 are of independent interest, since they also allow to get around the more difficult (and in general not available in the Heisenberg group) Hölder-type estimates.

We recall the standard notation of balls in the Heisenberg group (in fact, we deal with the so called Folland-Korányi balls, but the Carnot-Carathéodory balls would be good for our purposes too). For all $\xi := (z, t) \in \mathbb{R}^{2n} \times \mathbb{R}$, we define

$$\|\xi\|_{\mathbb{H}^n} := \sqrt[4]{|z|^4 + t^2}.$$

Then, for any $r > 0$, we set

$$\mathcal{B}_r := \{\xi \in \mathbb{R}^{2n+1} \text{ s.t. } \|\xi\|_{\mathbb{H}^n} < r\}.$$

We denote by \mathcal{L} the $(2n+1)$ -dimensional Lebesgue measure, and we observe that $\mathcal{L}(\mathcal{B}_r)$ equals, up to a multiplicative constant, to r^Q , where $Q := 2(n+1)$ is the homogeneous dimension of \mathbb{H}^n . Also, for all $g \in L^1(\mathcal{B}_r)$, we define the average of g in \mathcal{B}_r as

$$(g)_r := \frac{1}{\mathcal{L}(\mathcal{B}_r)} \int_{\mathcal{B}_r} g.$$

In what follows, we focus on L^p -estimates around a fixed point, say ξ_* , of Ω . Without loss of generality, we take ξ_* to be the origin, and we fix $R \in (0, 1)$ so small that $\mathcal{B}_R \Subset \Omega$.

Then, we denote by $\bar{u}_0 : \Omega \rightarrow \mathbb{R}$ the minimizer of problem (1) with $\varepsilon = 0$. Then, for a fixed $\varepsilon > 0$, we take $\bar{u}_\varepsilon : \mathcal{B}_R \rightarrow \mathbb{R}$ to be the minimizer of $\mathcal{F}_\varepsilon(u; \mathcal{B}_R)$ among all the functions $u \in W_{\mathbb{H}^n, 0}^{1,p}(\mathcal{B}_R)$, $u \leq \psi$, and $u - \bar{u}_0 \in W_{\mathbb{H}^n, 0}^{1,p}(\mathcal{B}_R)$. We can then extend \bar{u}_ε also on $\Omega \setminus \mathcal{B}_R$ by setting it equal to \bar{u}_0 in such a set. By construction

$$\begin{aligned} \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p &= \mathcal{F}_0(\bar{u}_0; \Omega) - \int_{\Omega \setminus \mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p \\ &\leq \mathcal{F}_0(\bar{u}_\varepsilon; \Omega) - \int_{\Omega \setminus \mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p = \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \end{aligned} \tag{20}$$

³It is worth pointing out that this is the only place in the paper where we use the condition that $p \in \mathcal{P}(\psi, \Omega)$. In particular, all the estimates in § 2 are valid for all $p \in (1, +\infty)$.

and

$$\begin{aligned} \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} &= \mathcal{F}_\varepsilon(\bar{u}_\varepsilon; \mathcal{B}_R) \\ &\leq \mathcal{F}_\varepsilon(\bar{u}_0; \mathcal{B}_R) = \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2}. \end{aligned} \quad (21)$$

Proposition 3. *Assume that*

$$p \in (1, 2]. \quad (22)$$

Then, there exists $C > 0$, only depending on n and p , such that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \leq C \left(1 + (|\nabla_{\mathbb{H}^n} \bar{u}_0|^p)_R\right)^{1-(p/2)} \varepsilon^{(p/2)^2} R^Q. \quad (23)$$

Proof. The technique for this proof is inspired by the one of Lemma 2.3 of [16], where a similar result was obtained in the quasilinear Euclidean case (however, our proof is self-contained). We have

$$\begin{aligned} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 &\leq (|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon| + |\nabla_{\mathbb{H}^n} \bar{u}_0|)^2 \\ &\leq C(|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2). \end{aligned} \quad (24)$$

Here, C is a positive constant, which is free to be different from line to line. By (22), (21) and (24), we obtain

$$\begin{aligned} &\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 \\ &\leq C \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \\ &= C \left(\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \right. \\ &\quad \left. + \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \right) \\ &\leq C \left(\int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} + \int_{\mathcal{B}_R} \frac{|\nabla_{\mathbb{H}^n} \bar{u}_0|^2}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{1-(p/2)}} \right) \\ &\leq C \left(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} + \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2} \right) \\ &\leq C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{p/2}. \end{aligned} \quad (25)$$

Thus, (25) and Lemma 9, applied here with $a := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $b := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$, yield

that

$$\begin{aligned}
& \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \\
& \leq C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0|^2 + \\
& \quad + C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} \\
& \leq C \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)}.
\end{aligned} \tag{26}$$

Now, from (20),

$$\begin{aligned}
& \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)} \\
& \leq \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \\
& \leq \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p.
\end{aligned} \tag{27}$$

Moreover, using (22) and some elementary calculus, we see that

$$|(1 + \tau)^{p/2} - \tau^{p/2}| \leq C$$

for any $\tau \geq 0$. Therefore, taking $\tau := \theta/\varepsilon$, we obtain that

$$|(\varepsilon + \theta)^{p/2} - \theta^{p/2}| \leq C\varepsilon^{p/2} \tag{28}$$

for any $\theta \geq 0$. Thus, using (27) and (28) with $\theta := |\nabla_{\mathbb{H}^n} \bar{u}_0|^2$, we conclude that

$$\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)} \leq C\varepsilon^{p/2} R^Q. \tag{29}$$

Now, we estimate the left hand side of (29) from below. For this scope, we define

$$\begin{aligned}
h &:= t\nabla_{\mathbb{H}^n} \bar{u}_0 + (1-t)\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon, \\
J &:= p \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \\
\text{and } \tilde{J} &:= p \int_{\mathcal{B}_R} \left[\int_0^1 (1-t) \frac{d}{dt} \left((\varepsilon + |h|^2)^{(p/2)-1} h \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \right) dt \right].
\end{aligned}$$

We observe that the variational inequality in (2) for \bar{u}_ε gives that

$$J \geq 0. \tag{30}$$

Also, using the Fundamental Theorem of Calculus, we obtain

$$\begin{aligned}
& \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)} \\
&= \int_{\mathcal{B}_R} \left[\int_0^1 \frac{d}{dt} (\varepsilon + |t \nabla_{\mathbb{H}^n} \bar{u}_0 + (1-t) \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)} dt \right] \\
&= p \int_{\mathcal{B}_R} \left[\int_0^1 (\varepsilon + |t \nabla_{\mathbb{H}^n} \bar{u}_0 + (1-t) \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \right. \\
&\quad \left. \times (t \nabla_{\mathbb{H}^n} \bar{u}_0 + (1-t) \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) dt \right] \\
&= p \int_{\mathcal{B}_R} \left[\int_0^1 (\varepsilon + |h|^2)^{(p/2)-1} h \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) dt \right].
\end{aligned}$$

Integrating by parts the latter integral in t (by writing $dt = \frac{d}{dt}(t-1) dt$), and exploiting (30), we obtain

$$\begin{aligned}
& \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2)^{(p/2)} - \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)} \\
&= J + \tilde{J} \geq \tilde{J}.
\end{aligned} \tag{31}$$

Making use of Lemma 8 – applied here with $a := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $b := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$ – we have that

$$\tilde{J} \geq \frac{1}{C} \int_{\mathcal{B}_R} \left[\int_0^1 (1-t) (\varepsilon + |t \nabla_{\mathbb{H}^n} \bar{u}_0 + (1-t) \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 dt \right].$$

From this and Lemma 10 – applied here with $\kappa := 1$ and $\Psi(x) := x^{1-(p/2)}$, which is nondecreasing, thanks to (22) – we deduce that

$$\tilde{J} \geq \frac{1}{C} \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2. \tag{32}$$

In view of (29), (31) and (32), we conclude that

$$\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_0|^2 + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2 \leq C \varepsilon^{p/2} R^Q. \tag{33}$$

Then, (23) follows from (26), (33) and Lemma 11, applied here with $f := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $g := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$. \square

In the degenerate case $p \in [2, +\infty)$ the estimate obtained in Proposition 3 for the singular case $p \in (1, 2]$ needs to be modified according to the following result:

Proposition 4. *Suppose that*

$$p \in [2, +\infty). \tag{34}$$

Then, there exists $C > 0$, only depending on n and p , such that

$$\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \leq C(1 + (|\nabla_{\mathbb{H}^n} \bar{u}_0|_R^p)^{1-(1/p)} \varepsilon R^Q).$$

Proof. The variational inequalities (2) for \bar{u}_0 and \bar{u}_ε imply that

$$\begin{aligned} & \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_0 \cdot (\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0) \geq 0 \\ \text{and} \quad & \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \geq 0. \end{aligned}$$

Consequently,

$$\int_{\mathcal{B}_R} \left(|\nabla_{\mathbb{H}^n} \bar{u}_0|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_0 - (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \leq 0.$$

Using this and (40) of Lemma 6, applied here with $A := \nabla_{\mathbb{H}^n} \bar{u}_0$ and $B := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$, we obtain

$$\begin{aligned} & \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \\ & \leq C \int_{\mathcal{B}_R} \left(|\nabla_{\mathbb{H}^n} \bar{u}_0|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_0 - |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon) \\ & \leq C \int_{\mathcal{B}_R} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon \right) \cdot (\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon). \end{aligned}$$

This and Corollary 7, applied here with $a := \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon$, give

$$\begin{aligned} & \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \\ & \leq C \int_{\mathcal{B}_R} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} - |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-2} \right) |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon| |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon| \\ & \leq C\varepsilon \int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p-2)/2} (|\nabla_{\mathbb{H}^n} \bar{u}_0| + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|). \end{aligned}$$

Therefore, recalling (34), noticing that

$$\frac{p-2}{p} + \frac{1}{p} + \frac{1}{p} = 1$$

and using the Generalized Hölder Inequality with the three exponents $p/(p-2)$, p and p , we obtain

$$\begin{aligned} & \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \\ & \leq C\varepsilon \left(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \right)^{(p-2)/p} \left(\int_{\mathcal{B}_R} (|\nabla_{\mathbb{H}^n} \bar{u}_0|^p + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p) \right)^{1/p} R^{Q/p}. \end{aligned}$$

Then, by the minimal property of \bar{u}_0 in (20),

$$\begin{aligned}
& \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0 - \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \\
& \leq C\varepsilon \left(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \right)^{(p-2)/p} \left(\int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \right)^{1/p} R^{Q/p} \\
& \leq C\varepsilon \left(\int_{\mathcal{B}_R} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{p/2} \right)^{(p-1)/p} R^{Q/p} \\
& \leq C\varepsilon \left(R^Q + \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^p \right)^{(p-1)/p} R^{Q/p} \\
& \leq C\varepsilon \left(R^Q + \int_{\mathcal{B}_R} |\nabla_{\mathbb{H}^n} \bar{u}_0|^p \right)^{(p-1)/p} R^{Q/p},
\end{aligned}$$

from which the desired result follows. \square

Corollary 5. *For all $p \in (1, +\infty)$, we have that*

$$\lim_{\varepsilon \searrow 0} \|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon - \nabla_{\mathbb{H}^n} \bar{u}_0\|_{L^p(\mathcal{B}_R)} = 0. \quad (35)$$

Also, there exist a subsequence of ε 's and a function $G \in L^p(\mathcal{B}_R)$ such that

$$|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon(x)| \leq G(x) \quad (36)$$

for almost every $x \in \mathcal{B}_R$.

Furthermore, if we set

$$\Gamma_\varepsilon := (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \bar{u}_\varepsilon, \quad (37)$$

then there exist a subsequence of ε 's and a function $G_\star \in L^1(\mathcal{B}_R)$ such that

$$|\Gamma_\varepsilon(x)| \leq G_\star(x) \quad (38)$$

for almost every $x \in \mathcal{B}_R$.

Proof. We obtain (35) from Propositions 3 and 4, according to whether $p \in (1, 2)$ or $p \in [2, +\infty)$.

From (35), one deduces (36) (see, e.g., Theorem 4.9(b) in [2]).

Now, we define $G_\star := 2^{(p/2)}(G + G^{p-1})$. We observe that $G_\star \in L^1(\mathcal{B}_R)$, since $G \in L^p(\mathcal{B}_R) \subseteq L^1(\mathcal{B}_R)$ and $G^{p-1} \in L^{p/(p-1)}(\mathcal{B}_R) \subseteq L^1(\mathcal{B}_R)$. So, in order to obtain the desired result, we have only to show that the inequality in (38) holds true.

For this, we notice that, if $p \in (1, 2)$,

$$\begin{aligned}
|\Gamma_\varepsilon| &= \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} = \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-1} |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{2-p}}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} \\
&\leq \frac{|\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-1} (\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{(2-p)/2}}{(\varepsilon + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^2)^{1-(p/2)}} = |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-1} \leq G^{p-1},
\end{aligned}$$

which implies (38) in this case.

On the other hand, if $p \in [2, +\infty)$,

$$\begin{aligned} |\Gamma_\varepsilon| &\leq 2^{(p/2)-1} (\varepsilon^{(p/2)-1} + |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon|^{p-2}) |\nabla_{\mathbb{H}^n} \bar{u}_\varepsilon| \\ &\leq 2^{(p/2)-1} (1 + G^{p-2}) G, \end{aligned}$$

which implies (38) in this case too. \square

3 Proof of Theorem 2 when $\varepsilon = 0$

By Theorem 2 (for $\varepsilon > 0$, which has been proved in § 1), we know that, for a sequence $\varepsilon \searrow 0$,

$$0 \leq \int_{\mathcal{B}_R} \Gamma_\varepsilon \cdot \nabla \varphi \leq \int_{\mathcal{B}_R} \left(\operatorname{div}_{\mathbb{H}^n} \left((\varepsilon + |\nabla_{\mathbb{H}^n} \psi|^2)^{(p/2)-1} \nabla_{\mathbb{H}^n} \psi \right) \right)^+ \varphi, \quad (39)$$

for any $\varphi \in C_0^\infty(\mathcal{B}_R, [0, +\infty))$, as long as $\mathcal{B}_R \subset \Omega$, where Γ_ε is as in (37).

By possibly taking subsequences, in the light of (35) and (38), we have that

$$\lim_{\varepsilon \searrow 0} \Gamma_\varepsilon = |\nabla_{\mathbb{H}^n} \bar{u}_0|^{p-2} \nabla_{\mathbb{H}^n} \bar{u}_0$$

almost everywhere in \mathcal{B}_R , and that Γ_ε is equidominated in $L^1(\mathcal{B}_R)$. Consequently, we can pass to the limit in (39) via the Dominated Convergence Theorem and obtain (6) for \bar{u}_0 . This completes the proof of Theorem 2 also when $\varepsilon = 0$. \square

Appendix

In this appendix, we collect some technical estimates of general interest that will be used in the proofs of the main results of this paper.

We start with some classical estimates (see, e.g. Lemma 3 in [9] and references therein), which turns out to be quite useful to deal with nonlinear operators of degenerate type:

Lemma 6. *Let $M \in \mathbb{N}$, $M \geq 1$, and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on M and p , such that, for any $A, B \in \mathbb{R}^M$,*

$$|A - B|^p \leq C \left(|A|^{p-2} A - |B|^{p-2} B \right) \cdot (A - B) \quad (40)$$

and

$$\left| |A|^{p-2} A - |B|^{p-2} B \right| \leq C |A - B| \left(|A|^{p-2} + |B|^{p-2} \right). \quad (41)$$

Corollary 7. *Let $N \in \mathbb{N}$ and $p \in [2, +\infty)$. Then, there exists $C > 1$, only depending on N and p , such that for any $\varepsilon > 0$ and any $a \in \mathbb{R}^N$*

$$((\varepsilon + |a|^2)^{(p/2)-1} - |a|^{p-2}) |a| \leq C \varepsilon (\varepsilon + |a|^2)^{(p-2)/2}.$$

Proof. We let $A := (a, \varepsilon)$ and $B := (a, 0) \in \mathbb{R}^{N+1}$ and we exploit (41). We obtain

$$\begin{aligned}
& 2C\varepsilon(\varepsilon + |a|^2)^{(p-2)/2} \\
& \geq C\varepsilon \left((\varepsilon + |a|^2)^{(p-2)/2} + |a|^{p-2} \right) \\
& = C|A - B| \left(|A|^{p-2} + |B|^{p-2} \right) \\
& \geq \left| |A|^{p-2}A - |B|^{p-2}B \right| \\
& = \left| (\varepsilon + |a|^2)^{(p-2)/2}(a, \varepsilon) - |a|^{p-2}(a, 0) \right| \\
& = \left| \left((\varepsilon + |a|^2)^{(p-2)/2} - |a|^{p-2} \right)a, (\varepsilon + |a|^2)^{(p-2)/2}\varepsilon \right| \\
& \geq \left((\varepsilon + |a|^2)^{(p-2)/2} - |a|^{p-2} \right) |a|,
\end{aligned}$$

as desired. \square

In the subsequent Lemmata 8 and 9, we collect some simple, but interesting, estimates that are used in Proposition 3:

Lemma 8. *Let $N \in \mathbb{N}$, $N \geq 1$, $t \in [0, 1]$, $\varepsilon > 0$, and $a, b \in \mathbb{R}^N$. Let $h(t) := ta + (1-t)b$. Then, there exists $C > 1$, only depending on N and p , such that*

$$\frac{d}{dt} \left((\varepsilon + |h|^2)^{(p/2)-1} h \cdot (a - b) \right) \geq \frac{1}{C} (\varepsilon + |ta + (1-t)b|^2)^{(p/2)-1} |a - b|^2.$$

Proof. We have

$$\begin{aligned}
\frac{d}{dt} \left((\varepsilon + |h|^2)^{(p/2)-1} h \cdot (a - b) \right) &= \frac{d}{dt} \left((\varepsilon + |h|^2)^{(p/2)-1} h \right) \cdot (a - b) \\
&= (\varepsilon + |h|^2)^{(p/2)-2} (\varepsilon + (p-1)|h|^2) \frac{dh}{dt} \cdot (a - b) \\
&\geq \frac{1}{C} (\varepsilon + |h|^2)^{(p/2)-1} |a - b|^2 \\
&= \frac{1}{C} (\varepsilon + |ta + (1-t)b|^2)^{(p/2)-1} |a - b|^2,
\end{aligned}$$

as desired. \square

Lemma 9. *Let*

$$p \in (1, 2]. \tag{42}$$

Let $N \in \mathbb{N}$, $N \geq 1$, $\varepsilon > 0$, and $a, b \in \mathbb{R}^N$. Then, there exists $C > 1$, only depending on N and p , such that

$$(\varepsilon + |a|^2 + |b|^2)^{p/2} \leq C \left[(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1} |b - a|^2 + (\varepsilon + |a|^2)^{(p/2)} \right].$$

Proof. We have

$$|b|^2 = |b - a + a|^2 \leq (|b - a| + |a|)^2 \leq C(|b - a|^2 + |a|^2)$$

and so

$$\begin{aligned} & (\varepsilon + |a|^2 + |b|^2)^{p/2} \\ = & (\varepsilon + |a|^2 + |b|^2)^{(p/2)-1} (\varepsilon + |a|^2 + |b|^2) \\ \leq & C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1} (\varepsilon + |a|^2 + |b - a|^2) \\ = & C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1} |b - a|^2 + C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1} (\varepsilon + |a|^2). \end{aligned}$$

Therefore, by (42),

$$\begin{aligned} & (\varepsilon + |a|^2 + |b|^2)^{p/2} \\ & \leq C(\varepsilon + |a|^2 + |b|^2)^{(p/2)-1} |b - a|^2 + C(\varepsilon + |a|^2 + |b|^2)^{(p/2)}, \end{aligned}$$

that is the desired claim. \square

The following result deals with some technical estimates on monotone integrands.

Lemma 10. *Let $N \in \mathbb{N}$, $N \geq 1$. Let $\kappa \in \{0, 1\}$. Let $\varepsilon, \varepsilon' > 0$. Let $a, b \in \mathbb{R}^N$. Let $\Psi : [\varepsilon, +\infty) \rightarrow [\varepsilon', +\infty)$ be a measurable and nondecreasing function. Then*

$$\int_0^1 \frac{(1-t)^\kappa}{\Psi(\varepsilon + |ta + (1-t)b|^2)} dt \geq \frac{1}{2\Psi(\varepsilon + |a|^2 + |b|^2)}. \quad (43)$$

Proof. If $|a| \leq |b|$, for any $t \in [0, 1]$,

$$\begin{aligned} |ta + (1-t)b|^2 & \leq t^2|a|^2 + (1-t)^2|b|^2 + 2t(1-t)|a||b| \\ & \leq t^2|b|^2 + (1+t^2-2t)|b|^2 + 2t(1-t)|b|^2 = |b|^2. \end{aligned}$$

On the other hand, if $|a| \geq |b|$, for any $t \in [0, 1]$,

$$\begin{aligned} |ta + (1-t)b|^2 & \leq t^2|a|^2 + (1-t)^2|b|^2 + 2t(1-t)|a||b| \\ & \leq t^2|a|^2 + (1+t^2-2t)|a|^2 + 2t(1-t)|a|^2 = |a|^2. \end{aligned}$$

In any case,

$$\varepsilon + |ta + (1-t)b|^2 \leq \varepsilon + |a|^2 + |b|^2$$

and the claim follows from the monotonicity of Ψ . \square

The next is a useful Hölder/ L^p type estimate, that is exploited in Proposition 3.

Lemma 11. *Let $N \in \mathbb{N}$, $N \geq 1$. Let $f, g \in L^p(\mathcal{B}_R, \mathbb{R}^N)$. Suppose that*

$$p \in (1, 2]. \quad (44)$$

Then

$$\begin{aligned} & \int_{\mathcal{B}_R} |f - g|^p \\ & \leq \left(\int_{\mathcal{B}_R} (\varepsilon + |f|^2 + |g|^2)^{(p/2)-1} |f - g|^2 \right)^{p/2} \\ & \quad \times \left(\int_{\mathcal{B}_R} (\varepsilon + |f|^2 + |g|^2)^{p/2} \right)^{(2-p)/2}. \end{aligned}$$

Proof. We observe that

$$\begin{aligned} & |f - g|^p \\ & = \left[(\varepsilon + |f|^2 + |g|^2)^{(p/2)-1} |f - g|^2 \right]^{p/2} \left[(\varepsilon + |f|^2 + |g|^2)^{p/2} \right]^{(2-p)/2}, \end{aligned}$$

and so the desired result follows from the Hölder Inequality with exponents $2/p$ and $2/(2-p)$, which can be used here due to (44). \square

To end this paper, we remark that Definition 1 is always nonvoid (independently of ψ and Ω), in the sense that

Lemma 12. $2 \in \mathcal{P}(\psi, \Omega)$.

Proof. The functional in (4) when $p = 2$ boils down to

$$\int_{\Omega} \frac{1}{2} |\nabla_{\mathbb{H}^n} u(\xi)|^2 + F_k(u(\xi), \xi) d\xi, \quad (45)$$

up to an additive constant that does not play any role in the minimization. Hence, if u_k minimizes this functional, we have that

$$- \int_{\Omega} \nabla_{\mathbb{H}^n} u_k(\xi) \cdot \nabla_{\mathbb{H}^n} \varphi(\xi) d\xi = \int_{\Omega} \partial_r F_k(u_k(\xi), \xi) \varphi(\xi) d\xi$$

for any $\varphi \in C_0^\infty(\Omega)$.

Accordingly, if also u_k approaches some u_∞ uniformly in Ω , it follows that

$$\begin{aligned} \int_{\Omega} u_\infty \Delta_{\mathbb{H}^n} \varphi &= \lim_{k \rightarrow +\infty} \int_{\Omega} u_k \Delta_{\mathbb{H}^n} \varphi \\ &= \lim_{k \rightarrow +\infty} - \int_{\Omega} \nabla_{\mathbb{H}^n} u_k \cdot \nabla_{\mathbb{H}^n} \varphi = \lim_{k \rightarrow +\infty} \int_{\Omega} \partial_r F_k(u_k, \xi) \varphi \end{aligned} \quad (46)$$

for any $\varphi \in C_0^\infty(\Omega)$.

Also, from (3),

$$0 \leq \partial_r F_k \leq (\Delta_{\mathbb{H}^n} \psi)^+$$

and so (46) gives that

$$0 \leq \int_{\Omega} u_\infty \Delta_{\mathbb{H}^n} \varphi \leq \int_{\Omega} (\Delta_{\mathbb{H}^n} \psi)^+ \varphi \quad (47)$$

for any $\varphi \in C_0^\infty(\Omega, [0, +\infty))$.

On the other hand, since u_k is a minimizer for (45), we have that

$$\sup_{k \in \mathbb{N}} \|\nabla_{\mathbb{H}^n} u_k\|_{L^2(\Omega)} < +\infty$$

and so, up to a subsequence, we may suppose that $\nabla_{\mathbb{H}^n} u_k$ converges to some $\nu \in L^2(\Omega)$ weakly in $L^2(\Omega)$. It follows from the uniform convergence of u_k that

$$\begin{aligned} - \int_{\Omega} \nu \cdot \nabla_{\mathbb{H}^n} \varphi &= - \lim_{k \rightarrow +\infty} \int_{\Omega} \nabla_{\mathbb{H}^n} u_k \cdot \nabla_{\mathbb{H}^n} \varphi \\ &= \lim_{k \rightarrow +\infty} \int_{\Omega} u_k \Delta_{\mathbb{H}^n} \varphi = \int_{\Omega} u_{\infty} \Delta_{\mathbb{H}^n} \varphi \end{aligned}$$

for any $\varphi \in C_0^\infty(\Omega)$. That is, $\nabla_{\mathbb{H}^n} u_{\infty} = \nu$ in the sense of distributions, and so as a function. In particular, $\nabla_{\mathbb{H}^n} u_{\infty} \in L^2(\Omega)$, and therefore (47) yields that

$$0 \leq \int_{\Omega} \nabla_{\mathbb{H}^n} u_{\infty} \cdot \nabla_{\mathbb{H}^n} \varphi \leq \int_{\Omega} (\Delta_{\mathbb{H}^n} \psi)^+ \varphi,$$

for any $\varphi \in C_0^\infty(\Omega, [0, +\infty))$. This shows that u_{∞} satisfies (5), in the distributional sense, hence as a function. \square

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